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Solving the Mathematical Model of the Electrode Sheath in Symmetrically Driven RF Discharges

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A mathematical analysis is given for a self-consistent dynamic model for rf sheaths in the frequency range between the ion and electron plasma frequencies for arbitrary collision parameters and arbitrary rf sheath voltages. Based on this analysis, an algorithm is developed to solve the problem numerically in its full generality. Using the method that is developed, it is found that the rf conductivity current causes an asymmetrical behavior of the plasma-sheath interface and of the sheath characteristics, but has little effect on the rf discharge current/voltage characteristics.

1. INTRODUCTION

Radio frequency (rf) discharges are encountered in a large number of modern engineering technologies. Some of the most commonly used applications are plasma processing for manufacturing semiconductor chips, plasma chemistry, gaseous lasers, and rf light sources. For a successful design and optimization of plasma technology devices based on rf discharges, it is necessary to understand the specific properties of the basic rf discharge parameters. A capacitive rf discharge is a highly nonlinear phenomenon, mainly because of the presence of the rf sheaths that separate the rf electrodes from the plasma. In the majority of applications, the impedance of the rf sheath is much greater than that of the plasma, and the rf electrode sheaths have a strong influence on the rf discharge electrical characteristics. The physical processes in the rf sheaths directly influence the type and rate of surface reactions that occur in plasma processing reactors and are critical factors in efficiency and durability of plasma sources based on capacitive rf discharges.

Different models have been developed to describe the characteristics of the rf sheaths. A review of these models, their strengths and deficiencies, can be found in Ref. [1]. A comparison of different rf electrode sheath models can be found in Ref. [2]. A self-consistant hydrodynamic model for the planar rf sheaths of a symmetrically driven rf discharge was formulated in Ref. [3]; in Ref. [1], we generalized it. In the model, we assumed an rf discharge without emissive processes at the rf electrodes and we considered the case when the rf frequency and the gas pressure satisfy the plasma opacity condition [1]. We also chose the plasma density and the rf frequency sufficiently high so that the Debye radius and the amplitude of the electron oscillations at the plasma–sheath boundary are significantly less than the electrode gap. Under these assumptions we have studied the nonlinear dynamics of the rf sheath for a steplike electron profile at the moving plasma–sheath interface.[1] Figure 1 gives a qualitative representation of the immobile ion distribution in the sheath. Under the action of the rf field, the plasma–sheath interface oscillates between its two extreme positions, the maximal sheath width and the minimal sheath width. The position of the plasma–sheath interface at any given time t can therefore be described by a periodic function $x = S(t)$ that achieves its maximum S_1 at $t = t_1$ and its minimum S_2 at $t = t_2$. To the right of the moving interface (in the plasma), the plasma neutrality is preserved and the electron density is equal to the ion density. To the left of the moving interface (in the sheath), the electron density is equal to zero. Although from the physical point of view the model in Ref. [1] is the most justified and consistent, it leads to a complicated moving boundary problem that previously has had to be simplified in order to be solved. A number of such simplified approaches have appeared in the literature [4–6]. Mathematical simplifica-

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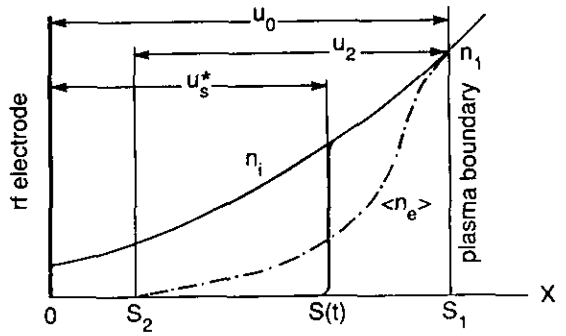

FIG. 1. Structure of the rf electrode sheath; n_i is the ion density distribution, $\langle n_e \rangle$ is the time-averaged electron density distribution, u_0 is the total sheath voltage, u_2 is the voltage at the minimal position of the plasma–sheath interface, and u_s is the voltage corresponding to the sheath space charge.

tions, however, can lead to physical inconsistencies and inaccuracies. In Ref. [1], we were able to solve the generalized model for arbitrary rf sheath voltages and collision parameters under the single restrictive assumption that the function S is symmetric with respect to its minimum position at $t = t_2$. Such an assumption is justified for heavy ions such as mercury. A comparison of the analytical and numerical solutions of the mathematical model with the available experimental results for mercury vapor have demonstrated a good agreement in corresponding values. The mathematical analysis, which underlies the numerical results, can be found in Ref. [7]. The mathematical method described in [7] solves the generalized model over a half period and symmetrically extends over a full period the solutions thus obtained. This method can be used only when the function S is symmetric with respect to its minimum position. In general, however, S is not symmetric because of the conductivity current in the rf sheath. The rf sheath conductivity current is proportional to the ion sound speed and is neglegibly small for the heavy mercury ions [1]. For lighter ions, such as argon, the neglect of the conductivity current might lead to inaccuracies.

Taking asymmetry into account requires new mathematical techniques for solving the moving boundary problem that describes our generalized model [1]. In this paper, we give a detailed mathematical analysis of the generalized model and develop an algorithm for solving the moving boundary problem numerically in its full generality. We then use the method that we develop to obtain the characteristics of the rf electrode sheath for argon rf discharges and to study the effect of the conductivity current on the sheath characteristics.

2. FORMULATION OF THE MATHEMATICAL MODEL

The dynamic model of the electrode sheath in symmetrically driven rf discharges in the frequency range between the ion and electron plasma frequencies for arbitrary collision parameters and arbitrary rf sheath voltages can be described by the following moving boundary hydrodynamic problem [1].

The instantaneous potential, V, in the sheath can be found from the Poisson equation

$$\frac{d^2V}{dx^2} = \begin{cases} -4\pi e n_i, & \text{for } x \leq S(t) \\ 0, & \text{for } x > S(t). \end{cases} \tag{1}$$

The stationary ion density distribution, n_i, can be found from the momentum-transfer equation

$$v_i \frac{dv_i}{dx} + \frac{e}{M}\left\langle \frac{dV}{dx} \right\rangle + \frac{eT_i}{Mn_i}\frac{dn_i}{dx} + \frac{F}{M} = 0 \tag{2}$$

and the continuity equation

$$\frac{d}{dx}(n_i v_i) = 0. \tag{3}$$

In Eqs. (1)–(3), e is the electron charge, v_i is the ion velocity, M is the ion mass, T_i is the ion temperature, F is the frictional force, and $\langle dV/dx \rangle$ is the electric field in the sheath averaged in time over an rf period. In what follows, we will retain the $\langle\ \rangle$ notation for values that are time averaged over an rf period. For a typical low pressure rf discharge, the frictional force is given by the formula

$$F = \frac{\pi M v_i^2}{2\lambda_i} \tag{4}$$

(see [3, p. 108]), where λ_i is the ion mean free path.

The equation describing the moving boundary S is obtained from the equation for the rf current I in the sheath, and by setting I equal to the rf current at the plasma boundary:

$$I = I_d + I_e + I_i = e n_1 v_{\sim 1} = e n_1 a_1 \omega, \tag{5}$$

where

$$I_e = e n_S \left[\frac{eT_e}{2\pi m}\right]^{1/2} \exp\left[\frac{V_0 - V_S}{T_e}\right] \tag{6}$$

is the conduction current of the electrons,

$$I_i = -e n_1 v_1 \tag{7}$$

is the conduction current of the ions, and

$$I_d = \frac{d}{dt}\int_0^{S(t)} e n_i\, dx = e n_S \frac{dS}{dt} + e\int_0^{S(t)} \frac{dn_i}{dt}\, dx \tag{8}$$

is the sheath displacement current. In the Eqs. (5)–(8), T_e is the electron temperature in volts, m is the electron mass, $v_\sim$ is the oscillatory electron velocity, a is the amplitude of the electron oscillations, ω is the rf frequency, and v is the ion velocity. Furthermore, index 0 indicates values at the electrode, index 1 corresponds to the values at $x = S_1$, and index S indicates values for $x = S(t)$. The term $V_0 - V_S$ gives the voltage between the rf electrode and the moving plasma–sheath interface. In order to solve the problem (1)–(8), we specify the rf discharge current

$$I(t) = I_0 \cos(\omega t) \tag{9}$$

and obtain the summarized rf sheath voltage V_c across the two sheaths [8]. Equations (1)–(8) with appropriate boundary conditions describe the dynamics of the rf sheath. The boundary conditions are determined by the parameters of the neutral plasma that is adjacent to the sheath at the point $x = S_1$. For convenience, we set $V_1 = 0$. The boundary between the plasma and the rf sheath is located at the point $x = S_1$, where the ambipolar dc electric field on the plasma side reaches the value

$$E_1 = T_e/\lambda_{D_1} \tag{10}$$

with λ_{D_1} the Debye radius at the plasma–sheath boundary [9]. Applied to the neutral plasma equations, this boundary condition yields the value of the ambipolar (ion) velocity

$$v_1 = \left[\frac{e(T_e + T_i)}{M}\right]^{1/2} \left[1 + \frac{\pi\lambda_{D_1}}{2\lambda_i}\right]^{-1/2}. \tag{11}$$

Observe that v_1 is the velocity with which ions are injected into the sheath [10]. For the collisionless sheath, $\lambda_{D_1}/\lambda_i \ll 1$ holds and formula (11) becomes the well-known Bohm criterion,

$$v_1 = \left[\frac{e(T_e + T_i)}{M}\right]^{1/2} = v_B, \tag{12}$$

where v_B is the ion sound speed.

The given problem can now be solved for one sheath, and since the instantaneous rf currents in both sheaths are equal, we can find the corresponding results for the other sheath by shifting the phase of the solution by $\omega t = \pi$.

The physical justifications and a more detailed description of the model (1)–(11) can be found in Ref. [1].

In order to solve the system (1)–(11), we introduce new dimensionless variables:

$$\xi = \frac{x}{\lambda_{D_1}}, \quad \theta = \omega t, \quad u = -\frac{V}{T_e}, \quad \eta = \langle u \rangle,$$
$$\lambda = \frac{S}{\lambda_{D_1}}, \quad y = \frac{n_i}{n_1}, \quad \rho = \frac{a_1}{\lambda_{D_1}}, \quad \sigma = \frac{v_B}{v_{\sim 1}}, \tag{13}$$
$$\alpha = \frac{\pi\lambda_{D_1}}{2\lambda_i}, \quad \tau = \frac{T_i}{T_e}, \quad \gamma = \left[\frac{M}{2\pi m}\right]^{1/2}.$$

Using the new variables (13) and assuming that $\tau = 0$, the system (1)–(11) can be reduced to the following system: the Poisson equation

$$\frac{d^2u(\xi, \theta)}{d\xi^2} = \begin{cases} y(\xi), & \text{for} \quad \xi \leqslant \lambda(\theta) \\ 0, & \text{for} \quad \xi > \lambda(\theta); \end{cases} \tag{14}$$

the equation for the ion distribution $y(\xi)$

$$\frac{d\langle u \rangle}{d\xi} + \frac{1}{(1+\alpha)\, y^3} \frac{dy}{d\xi} + \frac{\alpha}{(1+\alpha)\, y^2} = 0; \tag{15}$$

and the equation for the sheath current

$$\sigma[\gamma y(\lambda(\theta)) \exp[u(\lambda(\theta), \theta) - u(0, \theta)] - (1+\alpha)^{-1/2}] + \frac{y(\lambda(\theta))}{\rho} \frac{d\lambda}{d\theta} = \cos\theta. \tag{16}$$

Since the rf discharge is symmetric, the dc component of the current in each sheath must be equal to zero, i.e.,

$$\langle y(\lambda(\theta)) \exp[u(\lambda(\theta), \theta) - u(0, \theta)] \rangle = \gamma^{-1}(1+\alpha)^{-1/2}. \tag{17}$$

The conditions at the plasma–sheath boundary are

$$u(\lambda_1, \theta) = 0, \qquad \frac{du(\lambda_1, \theta)}{d\xi} = -1, \qquad y(\lambda_1) = 1. \tag{18}$$

It is given that λ is periodic.

There are four parameters in the problem: $\alpha \in [0, \infty]$ is the collision parameter, $\rho \geqslant 0$ is the oscillation parameter, which is proportional to the amplitude of the electron oscillations at the plasma–sheath boundary and describes the rf discharge current, $\sigma \ll 1$ is the sheath conductivity parameter and is proportional to the ion sound speed, and γ is a constant that depends on the mass of the gas (for instance, $\gamma = 242$ for mercury, $\gamma = 108$ for argon, and $\gamma = 34$ for helium).

Of primary interest is the dependence of the electrical sheath characteristics on the parameters. In this paper we concentrate mainly on the sheath conductivity parameter σ. It has been shown in Ref. [1] that for mercury vapor, the effect of σ on the sheath characteristics is negligibly small. This is due to the fact that for heavy ions, such as mercury, the solutions of the model are almost symmetric. The assumption of symmetry and the neglect of σ also greatly simplify the mathematical tools needed to solve the model. Whether or not σ can be neglected for light ions (such as argon), has up to now remained an open question. For light ions, one has to take into account the asymmetry of the solutions, and this requires new mathematical techniques in order to solve the model. In the next section these techniques will be developed, the system (14)–(18) will be solved in its full generality, and the question of whether or not σ can be neglected will be answered.

3. ANALYSIS

We make the following observations:

1. The function λ is 2π-periodic. Let T be the period of λ. Then, since y does not depend on θ, the function u is T-periodic in θ. Setting $d\lambda(\theta+T)/d\theta = d\lambda(\theta)/d\theta$, we obtain from Eq. (16) that $\cos\theta = \cos(\theta+T)$ and thus $T = 2\pi$.

2. Equation (16) implies that the point $\theta_1 \in [0, 2\pi]$, where λ achieves its maximum λ_1, can be obtained from the equation

$$\theta_1 = \arccos(\sigma\gamma \exp(-u(0, \theta_1)) - \sigma(1+\alpha)^{-1/2}). \quad (19)$$

3. The first two observations imply that Eq. (17) is equivalent to

$$\frac{1}{2\pi}\int_{\theta_1-2\pi}^{\theta_1} y(\lambda(\theta)) \exp[u(\lambda(\theta), \theta) - u(0, \theta)]\, d\theta = \gamma^{-1}(1+\alpha)^{-1/2}. \quad (20)$$

4. Consider the function λ on the interval $[\theta_1 - 2\pi, \theta_1]$. Let $\theta_2 \in [\theta_1 - 2\pi, \theta_1]$ be such that $\lambda(\theta_2) = \lambda_2$; i.e., λ achieves its minimum at θ_2. Then the function λ is strictly monotone decreasing from λ_1 to λ_2 on the interval $[\theta_1 - 2\pi, \theta_2]$ and strictly monotone increasing from λ_2 to λ_1 on the interval $[\theta_2, \theta_1]$. Continuity of λ implies that for each $\theta^* \in [\theta_2, \theta_1]$ there exists a unique $\theta_* \in [\theta_1 - 2\pi, \theta_2]$ such that $\lambda(\theta^*) = \lambda(\theta_*)$ (see Fig. 2). In physical terms, this means that within one time period, the plasma–sheath interface passes twice through every position between λ_1 and λ_2.

5. In order to find $\langle u \rangle$ one has to know the behavior of u as a function of θ. Let $\xi \in (\lambda_2, \lambda_1)$ be arbitrary. Then, by the previous observation, there exist $\theta^* \in (\theta_2, \theta_1)$ and $\theta_* \in (\theta_1 - 2\pi, \theta_2)$ such that $\lambda(\theta^*) = \lambda(\theta_*) = \xi$ and $\xi > \lambda(\theta)$ if and only if $\theta \in (\theta_*, \theta^*)$. Thus, for a fixed arbitrary $\xi \in [\lambda_2, \lambda_1]$,

$$\frac{d^2u(\xi, \theta)}{d\xi^2} = \begin{cases} y(\xi), & \text{for } \theta \in [\theta_1 - 2\pi, \theta_*] \cup [\theta^*, \theta_1] \\ 0, & \text{otherwise} \end{cases}$$

(see Fig. 3) and, therefore,

$$\frac{d^2\langle u \rangle}{d\xi^2} = \left\langle \frac{d^2u(\xi, \theta)}{d\xi^2} \right\rangle = \frac{1}{2\pi}\int_{\theta_1-2\pi}^{\theta_1} \frac{d^2u(\xi, \theta)}{d\xi^2}\, d\theta$$
$$= \frac{1}{2\pi}\int_{\theta_1-2\pi}^{\theta_*} y(\xi)\, d\theta + \frac{1}{2\pi}\int_{\theta^*}^{\theta_1} y(\xi)\, d\theta$$
$$= \frac{1}{2\pi}(2\pi + \theta_* - \theta^*)\, y(\xi). \quad (21)$$

For $\xi \in [0, \lambda_2]$, we obtain

$$\frac{d^2\langle u \rangle}{d\xi^2} = y(\xi). \quad (22)$$

In order to solve Eqs. (15) and (21) and to obtain the functions y and $\langle u \rangle$, one has to know the values of θ_* and θ^* for each ξ. To find these values, we consider θ_* as a function of θ^*. Since $\lambda(\theta_*) = \lambda(\theta^*)$, the function $\theta_*(\theta^*)$ satisfies the ordinary differential equation

$$\frac{d\theta_*}{d\theta^*} = \frac{d\lambda}{d\theta^*}\frac{d\theta_*}{d\lambda} \quad (23)$$

with the boundary condition $\theta_*(\theta_1) = \theta_1 - 2\pi$.

6. Consider Eq. (14). Integrating this equation for $\xi > \lambda(\theta)$ with the boundary condition (18), we find that $du(\xi, \theta)/d\xi = -1$ and $u(\xi, \theta) = \lambda_1 - \xi$ for all $\xi > \lambda(\theta)$ and $\theta \in [\theta_1 - 2\pi, \theta_1]$. Continuity of both u and $du/d\xi$ imply that $u(\lambda(\theta), \theta) = \lambda_1 - \lambda(\theta)$ and $du(\lambda(\theta), \theta)/d\xi = -1$. From now on, we will let $v(\xi, \theta) = du(\xi, \theta)/d\xi$.

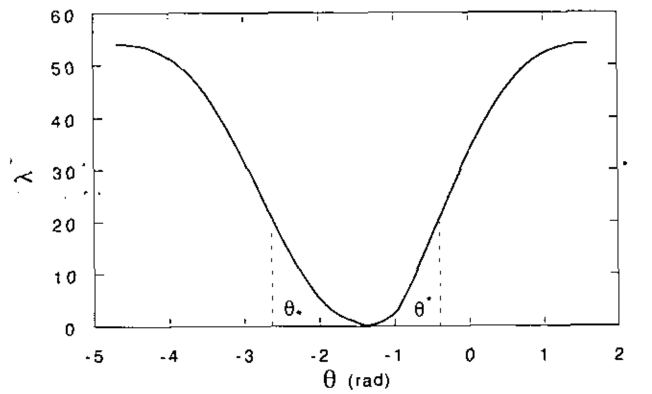

FIG. 2. Rf sheath width λ versus the phase $\theta = \omega t$ for $\sigma = 0.03$. The phases θ_* and θ^* correspond to the same value of λ, $\lambda(\theta_*) = \lambda(\theta^*)$.

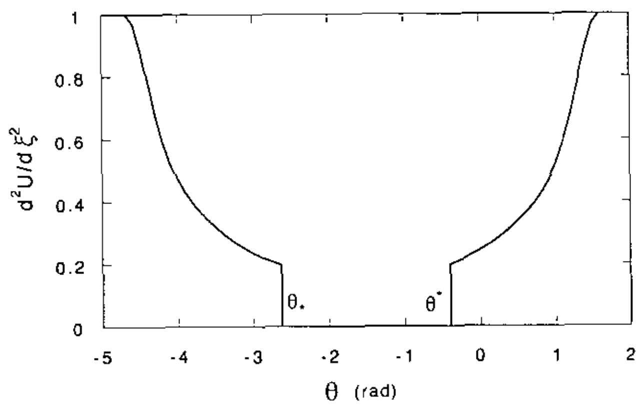

FIG. 3. Rf sheath space charge $d^2u(\xi, \theta)/d\xi^2$ versus the phase θ. Note that $d^2u(\xi, \theta)/d\xi^2 = 0$ for $\theta_* < \theta < \theta^*$.

7. Integrating Eq. (14) for $0 \leqslant \xi \leqslant \lambda(\theta)$, we obtain for all $\theta \in [\theta_1 - 2\pi, \theta_1]$:

$$\begin{aligned} v(\xi, \theta) &= -1 - \int_{\xi}^{\lambda(\theta)} y(\psi)\, d\psi \\ &= -1 - \int_{\xi}^{\lambda_1} y(\psi)\, d\psi + \int_{\lambda(\theta)}^{\lambda_1} y(\psi)\, d\psi \\ &= v(\xi, \theta_1) - v(\lambda(\theta), \theta_1) - 1. \end{aligned} \tag{24}$$

Integrating Eq. (14) once again using (24),

$$\begin{aligned} u(\xi, \theta) &= \lambda_1 - \lambda(\theta) - \int_{\xi}^{\lambda(\theta)} v(\psi, \theta)\, d\psi \\ &= \lambda_1 - \lambda(\theta) - \int_{\xi}^{\lambda(\theta)} v(\psi, \theta_1)\, d\psi \\ &\quad + \int_{\xi}^{\lambda(\theta)} (v(\lambda(\theta), \theta_1) + 1)\, d\psi \\ &= \lambda_1 - \lambda(\theta) - \int_{\xi}^{\lambda_1} v(\psi, \theta_1)\, d\psi + \int_{\lambda(\theta)}^{\lambda_1} v(\psi, \theta_1)\, d\psi \\ &\quad + [v(\lambda(\theta), \theta_1) + 1][\lambda(\theta) - \xi] \\ &= \lambda_1 - \xi + v(\lambda(\theta), \theta_1)[\lambda(\theta) - \xi] \\ &\quad + u(\xi, \theta_1) - u(\lambda(\theta), \theta_1). \end{aligned} \tag{25}$$

In particular,

$$v(0, \theta) - v(\lambda_2, \theta) = v(0, \theta_1) - v(\lambda_2, \theta_1) \tag{26}$$

and

$$\begin{aligned} u(0, \theta) - u(\lambda_2, \theta) &= u(0, \theta_1) - u(\lambda_2, \theta_1) \\ &\quad + \lambda_2[v(\lambda_2, \theta_1) - v(\lambda_2, \theta)]. \end{aligned} \tag{27}$$

Thus it is sufficient to solve the given system for $\theta = \theta_1$ and then use formulas (24) and (25) in order to obtain the solutions for all θ.

8. Observation 7 implies that $u(\xi, \theta_*) = u(\xi, \theta^*)$ and $v(\xi, \theta_*) = v(\xi, \theta^*)$ if $\theta_*, \theta^* \in [\theta_1 - 2\pi, \theta_1]$ are such that $\lambda(\theta_*) = \lambda(\theta^*)$.

The observations given above show that in order to find a solution to the given problem, we have to consider two regions between the plasma and the electrode. In the first region, $\lambda_2 \leqslant \xi \leqslant \lambda_1$ and the given problem is described by Eqs. (14)–(16), (21), and (23) with the boundary conditions (18), together with $d\langle u\rangle/d\xi = -1$ and $\theta_*(\theta_1) = \theta_1 - 2\pi$. In the second region, $0 \leqslant \xi \leqslant \lambda_2$, and the given problem is described by Eqs. (14)–(16) and (22) with the corresponding boundary conditions at $\xi = \lambda_2$.

9. In order to solve the problem in the region $\lambda_2 \leqslant \xi \leqslant \lambda_1$, we introduce a new variable $\varphi \in [\theta_1 - 2\pi, \theta_1]$, consider $\xi = \xi(\varphi)$, and let $s(\varphi) = \xi(\varphi) - \lambda_1$ be the solution of the equation

$$\begin{aligned} \frac{ds}{d\varphi} &= \frac{\rho \cos \varphi}{y(s(\varphi) + \lambda_1)} + \frac{\rho\sigma(1+\alpha)^{-1/2}}{y(s(\varphi) + \lambda_1)} \\ &\quad - \rho\sigma\gamma \exp[-s(\varphi) - u(0, \theta)] \end{aligned} \tag{28}$$

with the initial condition $s(\theta_1) = 0$. It follows from Eq. (16) and observation 6 that $s(\theta) = \lambda(\theta) - \lambda_1$. Using parameterization (28), we introduce new variables $\tilde{y}(\varphi) = y(\xi)$, $\tilde{u}(\varphi, \theta) = u(\xi, \theta)$, $\tilde{v}(\varphi, \theta) = v(\xi, \theta)$, $\tilde{\eta}(\varphi) = \langle u(\xi)\rangle = \eta(\xi)$, $\tilde{w}(\varphi) = d\langle u(\xi)\rangle/d\xi = w(\xi)$. With these new variables, since $ds/d\varphi = d\xi/d\varphi$, the given problem is equivalent to the system (with the tildas omitted in Eqs. (29)–(36)):

$$\frac{du(\varphi, \theta)}{d\varphi} = v(\varphi, \theta)\frac{ds}{d\varphi} \tag{29}$$

$$\frac{dv(\varphi, \theta)}{d\varphi} = \begin{cases} y(\varphi)(ds/d\varphi), \\ \quad \text{for} \quad \theta \in [\theta_1 - 2\pi, \theta_*] \cup [\theta^*, \theta_1] \\ 0, \quad \text{otherwise} \end{cases} \tag{30}$$

$$\frac{d\eta}{d\varphi} = w(\varphi)\frac{ds}{d\varphi} \tag{31}$$

$$\frac{dw}{d\varphi} = \frac{1}{2\pi}\left[2\pi + \left[\operatorname{sign}\left(\frac{ds}{d\varphi}\right)\right](\psi - \varphi)\right] y(\varphi)\frac{ds}{d\varphi} \tag{32}$$

$$\frac{dy}{d\varphi} = [-\alpha y(\varphi) - (1+\alpha)\, y^3(\varphi)\, w(\varphi)]\frac{ds}{d\varphi} \tag{33}$$

$$\begin{aligned} \frac{ds}{d\varphi} &= \frac{\rho \cos \varphi}{y(\varphi)} + \frac{\rho\sigma(1+\alpha)^{-1/2}}{y(\varphi)} \\ &\quad - \rho\sigma\gamma \exp[-s(\varphi) - u(0, \theta)] \end{aligned} \tag{34}$$

$$\frac{d\psi}{d\varphi} = \frac{\begin{bmatrix} \cos \varphi + \sigma(1+\alpha)^{-1/2} - \sigma\gamma y(\varphi) \\ \times \exp[-u(0, \varphi) - s(\varphi)] \end{bmatrix}}{\begin{bmatrix} \cos \psi + \sigma(1+\alpha)^{-1/2} - \sigma\gamma y(\psi) \\ \times \exp[-u(0, \psi) - s(\psi)] \end{bmatrix}}, \tag{35}$$

where $\theta_* = \psi$ and $\theta^* = \varphi$ if $\varphi \in [\theta_2, \theta_1]$, and $\theta_* = \varphi$ and $\theta^* = \psi$ if $\varphi \in [\theta_1 - 2\pi, \theta_2]$. Note that $s(\psi) = s(\varphi)$, $y(\psi) = y(\varphi)$, and by observation 8, $u(0, \varphi) = u(0, \psi)$. Furthermore, observation 7 implies that it is sufficient to solve system (29)–(35) for $\varphi \in [\theta_2, \theta_1]$, i.e., for $ds/d\varphi > 0$ and $\operatorname{sign}(ds/d\varphi) = 1$. Observation 7 also implies that it is sufficient to solve the system given above for $\theta = \theta_1$, and to use formulas (24) and (25) to obtain the solution for all θ. Thus the system above reduces to a system of ordinary differential equations which can be solved numerically with the initial conditions

$$\begin{gathered} u(\theta_1, \theta) = 0, \qquad v(\theta_1, \theta) = -1, \qquad \eta(\theta_1) = 0, \\ w(\theta_1) = -1, \qquad s(\theta_1) = 0, \qquad \psi(\theta_1) = \theta_1 - 2\pi, \end{gathered} \tag{36}$$

provided that $u(0, \varphi)$ is known. One then obtains the values of the solution at $\varphi = \theta_2$, which correspond to the values of the original problem at $\xi = \lambda_2$.

Note that the introduction of the variable $s(\varphi)$ enabled us to describe the moving boundary by a straight line $\theta = \varphi$, rather than by an unknown curve $\lambda(\theta)$ (see also Ref. [7]).

10. Consider the region $0 \leqslant \xi \leqslant \lambda_2$. In this region, the given problem is equivalent to the system:

$$\frac{du(\xi, \theta)}{d\xi} = v(\xi, \theta) \tag{37}$$

$$\frac{dv(\xi, \theta)}{d\xi} = y(\xi) \tag{38}$$

$$\frac{d\eta}{d\xi} = w(\xi) \tag{39}$$

$$\frac{dw}{d\xi} = y(\xi) \tag{40}$$

$$\frac{dy}{d\xi} = -\alpha y(\xi) - (1+\alpha)\, y^3(\xi)\, w(\xi), \tag{41}$$

where η and ω are as in the previous observation. As before, it is sufficient to solve this system for $\theta = \theta_1$ and to use formulas (24) and (25) to obtain the solution for all θ. Thus system (37)–(41) reduces to a system of ordinary differential equations, which can be solved using the values of the solution at $\xi = \lambda_2$ as initial conditions, provided that λ_2 is known.

11. In order to solve system (29)–(35) and system (37)–(41), we find the values of λ_2 and $u(0, \theta)$ through an iteration method using Eq. (20), which in the new coordinates becomes

$$\frac{1}{2\pi} \int_{\theta_1 - 2\pi}^{\theta_1} y(\theta) \exp[-s(\theta) - u(0, \theta)]\, d\theta = \gamma^{-1}(1+\alpha)^{-1/2}. \tag{42}$$

4. RESULTS AND DISCUSSION

The foregoing analysis has enabled us to obtain a complete numerical solution for the given problem without resorting to any restrictive assumptions. The calculations were performed for a symmetrically driven rf discharge in argon with moderate collisionality and the sheath conductivity parameter $\sigma = 0.03$ which is typical in experiments. The value of the collision parameter α varies in the experiment for constant gas pressures [11]. In order to obtain results in a form that is more convenient for applications, we have chosen a different collision parameter $\beta = 2\alpha\rho/\pi = a_1/\lambda_i$, and we have considered the case $\beta = 1$. In the experiment, the collision parameter β remained constant for constant gas pressures. In order to emphasize the nonlinearity caused by the asymmetrical behavior of the plasma–sheath interface, $\lambda(\theta)$, we have considered values for the oscillation parameter up to $\rho = 7$. This maximal value of ρ corresponds to rf sheath voltages of 1–2 kV, the maximal voltages encountered in experiments. Figure 2 shows a small asymmetry in the function $\lambda(\theta)$ with respect to its minimal position at $\theta \approx -1.36$ for $\sigma = 0.03$. This asymmetry leads to corresponding asymmetries in the soltions to the problem, in particular in the summarized rf sheath voltage

$$u_c = u(0, \theta) - u(0, \theta + \pi). \tag{43}$$

For $\sigma = 0.03$, the normalized waveforms of the summarized rf sheath voltage $u_c/\max(u_c)$ are given in Fig. 4 for small ($\rho = 0.3$) and large ($\rho = 7$) rf sheath voltages. As one can see, the behavior of u_c is almost symmetric for small rf sheath voltages. For large rf sheath voltages, the asymmetric behavior of u_c is more pronounced. From the practical point of view, it is important to know the effect of asymmetry on the integral characteristic of the sheath which governs the current/voltage characteristics of the rf discharge. In capacitive rf discharges, the current/voltage characteristic is almost entirely determined by the rf sheath capacitance

$$J = C\omega V_{\text{sh}}, \tag{44}$$

where J is the rf discharge current density, ω is the driving frequency, and C is the sheath capacitance per unit of the discharge cross section. Moreover, $C = (8\pi\lambda_0\lambda_{D_1})^{-1}$, where λ_0 is the capacitive (averaged) rf sheath thickness [1]. Figure 5 demonstrates the dependence of λ_0 on the amplitude of the summarized rf sheath voltage u_c^1 for the symmetric ($\sigma = 0$) and the asymmetric ($\sigma = 0.03$) cases. As one can see, even for very large rf sheath voltages ($u_c^1 \approx 700$), the difference in the values for λ_0 in both cases is negligibly

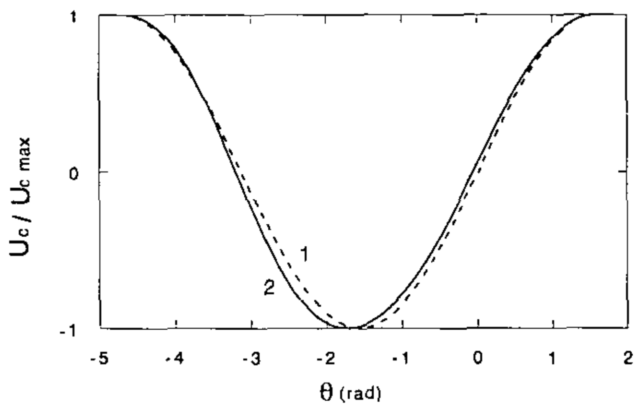

FIG. 4. Waveforms of the summarized rf sheath voltage for $\sigma = 0.03$: label 1 corresponds to small rf sheath oscillations ($\rho = 0.3$); label 2 corresponds to large rf sheath oscillations ($\rho = 7$).

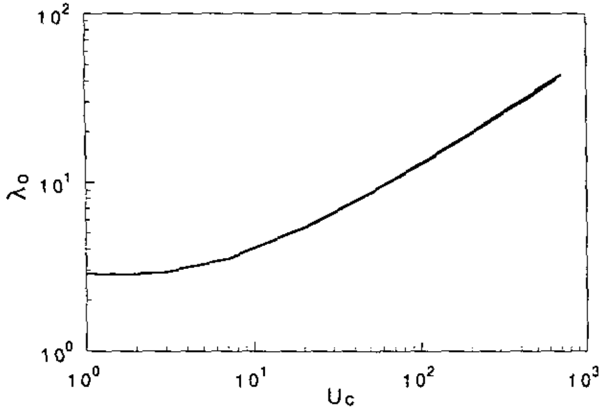

FIG. 5. Capacitive rf sheath width λ_0 versus the rf sheath voltage u_c^1 for $\sigma = 0$ (upper curve) and $\sigma = 0.03$ (lower curve). Note that the two curves almost coincide.

small. Thus the rf sheath capacitance is independent of the rf sheath conductivity current. In obtaining the rf discharge current/voltage characteristic, one can therefore neglect the sheath conductivity current and simplify the computations by setting $\sigma = 0$.

5. CONCLUSION

The rf sheath conductivity current causes an asymmetrical behavior of the plasma–sheath interface with respect to its minimum position. This asymmetry leads to corresponding asymmetries in the sheath characteristics, particularly in the summarized rf sheath voltage. The asymmetric behavior is more pronounced for large rf sheath voltages. The rf sheath capacitance, however, is almost independent of the rf sheath conductivity current. In obtaining the rf discharge current/voltage characteristic, the rf sheath conductivity current can therefore be neglected.

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